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Note

More mutually disjoint Steiner systems $S(5, 8, 24)$

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Abstract

Kramer and Magliveras constructed simple 5-$(24, 8, \lambda)$ designs for $\lambda \leqslant 9$. Betten et al. constructed simple 5-$(24, 8, \lambda)$ designs for $\lambda = 16, 17, \ldots, 484$. In this paper, we construct 15 mutually disjoint Steiner systems $S(5, 8, 24)$ and thus we construct simple 5-$(24, 8, \lambda)$ design for $\lambda = 10, 11, 12, 13, 14$ and 15. As a consequence there exists a simple 5-$(24, 8, \lambda)$ design for any positive integer $\lambda \leqslant \lambda_{\max} = 969$.

1. Introduction

Let $\mathcal{D} = (X, \mathcal{B})$ be a t-(v, k, λ) design with a set X of points and a set $\mathcal{B}$ of blocks where $X = \{1, 2, \ldots, v\}$ and $\mathcal{B} \subset \binom{X}{k} := \{B \subset X \mid |B| = k\}$ is a collection of k-subsets of X. A t-$(v, k, 1)$ design is called a Steiner system and denoted by $S(t, k, v)$. In this paper, all designs are simple, i.e., they have no repeated blocks.

The well-known Steiner system $S(5, 8, 24)$ is constructed by taking as blocks the supports of the minimal weight codewords of the extended binary Golay code. This design is called the Witt system and its automorphism group is the Mathieu group M_{24}. In [3], Kramer and Magliveras constructed 5-$(24, 8, \lambda)$ designs for $\lambda \leqslant 9$ by finding 9 mutually disjoint $S(5, 8, 24)$. Betten et al. constructed numerous designs using their software DISCRETA which includes an algorithm based on the methods of Kramer and Mesner [4]. In [1], Betten et al. showed that there exist 5-$(24, 8, \lambda)$ designs with automorphism group $PSL(2, 23)$ or $PGL(2, 23)$ for $\lambda = 16, 17, \ldots, 484$.

In this paper we show the following theorem.

Theorem 1. *There exists at least* 15 *mutually disjoint Steiner systems* $S(5, 8, 24)$.

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Suppose that $\mathscr{D}_1 = (X, \mathscr{B}_1), \mathscr{D}_2 = (X, \mathscr{B}_2), \ldots, \mathscr{D}_n = (X, \mathscr{B}_n)$ are mutually disjoint t-(v, k, λ) designs, i.e., $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_n \subset \binom{X}{k}$ and $\mathscr{B}_i \cap \mathscr{B}_j = \phi$ $(i \neq j)$. Then the union $\mathscr{D} = (X, \mathscr{B}_{i_1} \cup \mathscr{B}_{i_2} \cup \cdots \cup \mathscr{B}_{i_m})$ of these designs is a t-$(v, k, m\lambda)$ design for all $m = 1, 2, \ldots, n$ and $\{i_1, i_2, \ldots, i_m\} \in \binom{\{1,2,\ldots,n\}}{m}$. Let $\lambda_{\max} = \binom{v-t}{k-t}$. If $\mathscr{D} = (X, \mathscr{B})$ is a t-(v, k, λ) design, then $\bar{\mathscr{D}} = (X, \binom{X}{k} - \mathscr{B})$ is a t-$(v, k, \lambda_{max} - \lambda)$ design. By the above theorem and results of Kramer and Magliveras [3] and Betten et al. [1], we obtain the following corollary.

Corollary 2. *There exists a* 5-$(24, 8, \lambda)$ *design for any positive integer* λ *with* $\lambda \leqslant \lambda_{\max} = 969$.

2. Construction

Let $G = PSL(2, 23)$ be a permutation group on $X = \{\infty\} \cup \{0, 1, 2, \ldots, 22\}$ generated by two permutations

$$\alpha := (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22)$$

and

$$\gamma := (\infty, 0)(15, 3)(7, 13)(14, 18)(5, 9)(10, 16)(20, 8)(17, 4)(11, 2)(22, 1)(21, 12)(19, 6).$$

Then the group G acts on $\binom{X}{8}$ and the G-orbit $\mathscr{B}$ of the 8-set $\{\infty, 0, 1, 3, 12, 15, 21, 22\}$ forms the block set of a $S(5, 8, 24)$.

Our computer search shows that the set of block sets $\{\mathscr{B}, \mathscr{B}^{\sigma_1}, \ldots, \mathscr{B}^{\sigma_{14}}\}$ is mutually disjoint for the following 14 permutations,

$$\sigma_1 = (1, 6, 13, 14, \infty, 19, 22, 4, 12, 7, 20)(2, 17, 21, 10, 3, 18, 0, 15, 5, 8)(9, 16),$$

$$\sigma_2 = (1, 4, 16, 2)(5, 21, 9, 11, 15, 20, 12, \infty, 19, 13, 10, 0, 17, 8, 7, 22)(6, 18),$$

$$\sigma_3 = (1, 14, 7, 16, 20, 0, 18, 5, 19, 6, 4, 22, 8, 11, 15, 2, 12, \infty, 10, 3, 17, 13, 9, 21),$$

$$\sigma_4 = (1, 2, 12, 20, 13, 9, 7, 11)(3, 21, 10, 14, 6, 4, 22, 0, 15, 8, 19, \infty, 16)(17, 18),$$

$$\sigma_5 = (1, 18, 8, 20, 19, 10, 11, 9, 3, 2, 5, 16, 14, 4, 15, \infty, 13, 22, 17)(6, 7),$$

$$\sigma_6 = (1, 13, 4, 7, 19, 0, 15, 6, 20, 9, 2, 22, 16, 17, 8, 10, 12, 14, 21, 18, 5),$$

$$\sigma_7 = (1, 16, 3, 17, 2, 9, 7, 22, 15, 4, 20, 18, 13)(6, 19, 11, 10, 12, \infty, 14)(8, 21),$$

$$\sigma_8 = (1, 0, \infty, 17, 12, 4, 10, 21, 2, 19, 20, 9, 22, 11, 7, 6, 8, 14, 18, 16)(3, 15),$$

$$\sigma_9 = (1, 18, \infty, 21, 15, 8, 16, 6, 4, 14, 22, 17, 10, 7, 13, 5, 0, 9, 11, 19, 12, 3, 20, 2),$$

$$\sigma_{10} = (1, 9, 10, 4, 6, 13, 8, 19, 16, 22, 5, \infty, 18, 0, 21, 15, 2, 7, 20, 14, 17)(3, 12),$$

$$\sigma_{11} = (2, 0, 7, 9, 21)(3, 6, 5, 22, 17, 13, 8, 20, 14, 16, 18, 15, 10, 12, 4, 11, \infty, 19),$$

$$\sigma_{12} = (1, 18, 7, 21, 14, 22, 8, 0, 20, 17, \infty, 3, 5, 11, 15, 16, 2, 12, 10, 19, 6, 9, 13),$$

$$\sigma_{13} = (1, 20, 0, 22, 4, 18, 6, 2, 11, 10, 5, 21, 17, 7, 19, 16, 15, 13, 3, 14, \infty, 12, 9, 8)$$

and

$$\sigma_{14} = (1, 18, 16, 11, 20, 17)(2, 0, 14, 19, \infty, 10, 5, 9, 12, 4, 22, 7)(6, 13, 8, 21).$$

Hence there exists a 5-$(24, 8, \lambda)$ design for $\lambda \leqslant 15$.

Remark 3. We used the software GAP [2] and GRAPE [5] to find this design. By this result, we know that the maximal number of mutually disjoint $S(5, 8, 24)$'s is at least 15. We do not know whether there exists a large set of disjoint $S(5, 8, 24)$'s, i.e., a partition of the complete design into 969 disjoint $S(5, 8, 24)$'s.

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References

[1] A. Betten, E. Haberberger, R. Laue, A. Wassermann, DISCRETA—a program to construct t-designs with prescribed automorphism group, (http://btm2xl.mat.uni-bayreuth.de/discreta/).

[2] The GAP Group, GAP—Groups, Algorithms, and Programming, Version 4.3, 2002 (http://www.gap-system.org).

[3] E.S. Kramer, S.S. Magliveras, Some mutually disjoint Steiner systems, J. Combin. Theory A 17 (1974) 39–43.

[4] E.S. Kramer, D.M. Mesner, t-design on hypergraphs, Disc. Math. 15 (1976) 263–296.

[5] L.H. Soicher, GRAPE: a system for computing with graphs and groups, in: L. Finkelstein, W.M. Kantor (Eds.), Groups and Computation, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 11, 1993, pp. 287–291.